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RISE VELOCITY OF A SWARM OF SPHERICAL BUBBLES THROUGH A NON-NEWTONIAN FLUID: EFFECT OF ZERO SHEAR VISCOSITY

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Abstract—Happel's free surface cell model has been combined with the equations of continuity and motion for the creeping motion of an ensemble of spherical bubbles through a generalized Newtonian fluid. The resultant equations have been solved approximately, using the variational principles, and upper and lower bounds to the drag force experienced by a swarm of bubbles have been obtained. Wide ranges of gas contents and non-Newtonian fluid parameters have been covered in this study. A method for estimating the rise velocity of a swarm of bubbles is also presented.

INTRODUCTION

Dispersion of gases into liquids, a widespread phenomenon encountered in chemical and processing industries, commonly yields swarms of bubbles. Examples of such flow include antibiotic fermentation, wastewater treatment, polymer and food processing, gas–liquid reactions etc. The proximity of other bubbles alters the fluid streamlines around an individual bubble and thereby affects the rates of heat and mass transfer processes. It is often desirable to estimate the terminal velocity (or drag coefficient–Reynolds number relationship) of a swarm of bubbles rising through a pool of liquid.

Much work on the hydrodynamics of single bubbles (e.g. Nakano & Tien 1968; Hirose & Moo-Young 1969; Mohan 1974; Mohan & Venkateswarlu 1976; Acharya *et al.* 1977; Bhavaraju *et al.* 1978; Kawase & Ulbrecht 1981; Kawase & Moo-Young 1985; Chhabra & Dhingra 1986) has been reported. However, very little is known about the analogous situation of a swarm of bubbles. Gal-Or & Waslo (1968) employed Happel's free surface cell model (1958) to account for the interaction between bubbles moving in a Newtonian liquid. They obtained an analytical expression for the terminal velocity of an ensemble of spherical bubbles rising slowly through a pool of liquid. Marrucci (1965), considering the case of high Reynolds number, employed irrotational flow approximations.

Unfortunately, not all liquids of practical interest exhibit Newtonian behaviour. Indeed in many applications encountered in fermentation, polymer processing etc. (Bhavaraju & Blanch 1976; Astarita & Mashelkar 1977), the liquid phase displays rheologically complex behaviour including shear thinning, viscoelasticity, time dependency etc. The equations governing the motion of a swarm of bubbles through a non-Newtonian liquid phase are hopelessly complex due to the non-linear relation between stress and rate-of-strain tensor. Even when only shear thinning is considered, the pertinent equations are not amenable to rigorous analysis, and often approximate methods such as variational principles and perturbation techniques are used. Indeed, there have been, as far as we know, only two studies (Bhavaraju *et al.* 1978; Jarzebski & Malinowski 1986) relating to the hydrodynamic behaviour of a swarm of bubbles rising through power law liquids. Bhavaraju *et al.* (1978) employed a perturbation (around the Newtonian solution) scheme, whereas Jarzebski & Malinowski (1986) used variational principles to obtain approximate results for the terminal rise velocity of a swarm of monosized spherical bubbles rising through power law liquids. The power law model provides the simplest representation of shear thinning behaviour but its inability to predict a constant viscosity in the limit of low deformation rates (as exhibited by most materials) raises doubts about its suitability for describing creeping flows with stagnation points such as flow around a bubble. Thus the power law model does not accurately describe, at least, a finite part of the flow domain. This limitation of the power law has been convincingly demonstrated in the case of slow flow around a rigid sphere (Chhabra *et al.* 1980; Chhabra & Uhlherr 1981).

In this work a non-Newtonian fluid model containing zero shear viscosity (the limiting value of viscosity at low deformation rates) is used in conjunction with the equations of continuity and of motion to analyse the creeping motion of an ensemble of bubbles rising through the quiescent pool of a non-Newtonian liquid. Variational principles proposed by Slattery (1972) have been used to obtain upper and lower bounds on the drag coefficient of the swarm. The results reported herein can readily be converted into terminal velocity in the creeping flow regime.

PROBLEM STATEMENT

Consider the steady, creeping and axisymmetric flow of an incompressible non-Newtonian shear thinning liquid past an assemblage of monosized spherical gas bubbles (with a clean interface) with a superficial velocity U in the positive z direction, as shown in figure 1. Happel's free surface cell model postulates that each bubble (of radius R) is surrounded by a hypothetical spherical envelope whose (i) surface is frictionless, and (ii) radius is given by $R\epsilon^{-\frac{1}{3}}$ where ϵ is the overall average gas holdup (or content) of the dispersion. More detailed descriptions of the model are available in the literature (Happel 1958; Gal-Or & Waslo 1968).

The equations of continuity and of motion for creeping flow in compact notations may be written as follows:

$$V^i, i = 0 \qquad [1]$$

and

$$\tau^{ij}, j - p^i + \rho f^i = 0, \qquad [2]$$

where V^i, τ^{ij} and f^i are components of the velocity vector, extra stress tensor and body force, respectively; p is the isotropic pressure and ρ is the liquid density.

In a spherical coordinate system (r, θ, ϕ), the relevant boundary conditions are

$$\begin{aligned} v_r &= 0 \quad \text{at} \quad r = R, \\ \tau_{r\theta} &= 0 \quad \text{at} \quad r = R, \\ \tau_{r\theta} &= 0 \quad \text{at} \quad r = R\epsilon^{-\frac{1}{3}}, \\ v_r &= U\cos\theta \quad \text{at} \quad r = R\epsilon^{-\frac{1}{3}}. \end{aligned} \qquad [3]$$

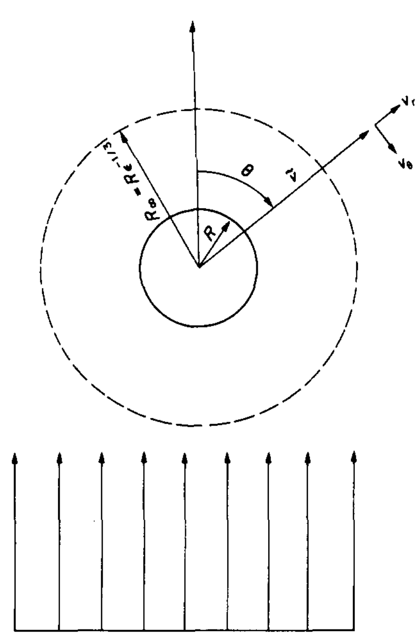

Figure 1. Schematic representation of the flow system.

We choose the Carreau (1972) viscosity equation to represent the shear thinning behaviour of the liquid phase. The Carreau viscosity equation not only describes the experimental data of shear stress (or apparent viscosity)/shear rate for a variety of materials remarkably well but has also yielded useful results in similar hydrodynamic situations (Chhabra & Uhlherr 1980; Chhabra & Raman 1984; Chhabra & Dhingra 1986). In steady shear it is written as

$$\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = (1 + 2\lambda^2 \mathrm{II})^{\frac{n-1}{2}}, \qquad [4]$$

where η is the apparent viscosity; η_0 and η_∞ are the zero shear and infinite shear viscosities, respectively; λ is a time parameter of the liquid; n is the slope of shear stress/shear rate data in the shear thinning region, and 2II is the second invariant of the rate-of-deformation tensor. For shear thinning materials, $n < 1$. For most materials, usually $n_\infty <<< n_0$, as suggested by Abdel-Khalik *et al.* (1974) and demonstrated by Boger (1977). Furthermore, the value of η_∞ is reached at very high shear rates that are unlikely to be encountered in the present case, where only creeping flow is being considered. Thus [4] is re-written as

$$\eta = \eta_0 (1 + 2\lambda^2 \mathrm{II})^{\frac{n-1}{2}}. \qquad [5]$$

Due to symmetry in the ϕ direction, the flow is two-dimensional, whence a stream function Ψ can be defined such that the equation of continuity is automatically satisfied. The two non-zero components of the velocity vector are related to the stream function as

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}$$

and

$$v_\theta = \frac{1}{r \sin\theta}\frac{\partial\psi}{\partial r}. \qquad [6]$$

The simultaneous solution of [1]–[6] would yield expressions for stream function, stress and pressure fields which, in turn, can be used to derive integrated quantities such as drag force.

ANALYSIS

Slattery (1972) presented the following two variational principles for the creeping and steady flow of an incompressible fluid:

velocity variational principle

$$J_V = \int_V E_1^* \,\mathrm{d}V + \int_{S-S_V} (\mathbf{V} - \mathbf{V}^*)\cdot([\mathbf{T} - \rho\phi\mathbf{I}]\cdot\mathbf{n})\,\mathrm{d}S; \qquad [7]$$

stress variational principle

$$H_\tau = -\int_V E_c^* \,\mathrm{d}V + \int_{S_\tau} \mathbf{V}\cdot([\mathbf{T} - \rho\phi\mathbf{I}]^*\cdot\mathbf{n})\,\mathrm{d}S; \qquad [8]$$

where the quantities with an asterisk appearing in [7] are evaluated from a trial stream function that satisfies the equation of continuity and the appropriate boundary conditions on S_V; the latter being the part of the bounding surface (S) on which the velocity is explicitly specified. Likewise, the quantities with an asterisk appearing in [8] are obtained from a trial extra stress profile that satisfies Cauchy's first law and the prescribed boundary condition for stress on S_τ. Chhabra & Raman (1984) showed that for the flow of a Carreau model fluid:

$$2J_V \geqslant \left[\int_V \mathrm{tr}(\boldsymbol{\tau}\cdot\nabla\mathbf{V})\,\mathrm{d}V\right] \geqslant (n+1)H_\tau. \qquad [9]$$

Thus the functionals J_V and H_τ provide upper and lower bounds on the quantity

$$\left[\int_V \mathrm{tr}(\boldsymbol{\tau}\cdot\nabla\mathbf{V})\,\mathrm{d}V\right],$$

which is the rate of energy dissipation per unit volume of the flow system. In this formulation, however, the contribution of the dispersed phase to the energy dissipation has been neglected. E_1 and E_c appearing in [7] and [8] are known as work and complementary work functions respectively, and for a Carreau model fluid are given by (Chhabra & Raman 1984)

$$E_1 = \frac{\eta_0}{(n+1)\lambda^2}[(1+2\lambda^2 \mathrm{II})^{\frac{n+1}{2}} - 1] \tag{10}$$

and

$$E_c = 2\eta_0(1+2\lambda^2\mathrm{II})^{\frac{n-1}{2}}\cdot \mathrm{II} - \frac{\eta_0}{(n+1)\lambda^2}[(1+2\lambda^2\mathrm{II})^{\frac{n+1}{2}} - 1]. \tag{11}$$

The usefulness of these variational principles in obtaining approximate solutions to similar hydrodynamic problems involving the creeping flow of non-Newtonian fluids has been illustrated in a number of papers, e.g. Mohan & Venkateswarlu (1976), Chhabra & Uhlherr (1980), Chhabra & Raman (1984), Chhabra & Dhingra (1986) and Cho & Hartnett (1983).

UPPER BOUND CALCULATIONS

The calculation of upper bound on the drag force experienced by a swarm of bubbles requires the knowledge of a trial stream function. The following trial stream function in dimensionless form, previously used for the slow flow past an assemblage of rigid spheres (Chhabra & Raman 1984), is adopted here:

$$\bar{\psi} = \sin^2\theta\left(\mathrm{A}_1 x^2 + \mathrm{A}_2 x^\sigma + \frac{\mathrm{A}_3}{x} + \mathrm{A}_4 x^4\right), \tag{12}$$

where $\bar{\psi}$ is the dimensionless stream function, A_1–A_4 are four constants which are to be evaluated using the boundary conditions given by [3] and x is the dimensionless radial coordinate defined as r/R. The four constants A_1–A_4 are determined as

$$\mathrm{A}_2 = \frac{3}{(\sigma+1)(\sigma-4)(\epsilon^{\frac{2-\sigma}{3}}-1)},$$

$$\mathrm{A}_1 = \frac{\mathrm{A}_2}{6}(\sigma+1)(\sigma-4),$$

$$\mathrm{A}_3 = \frac{\mathrm{A}_2}{6}\frac{(\sigma-1)(\sigma-2)}{(\epsilon^{\frac{5}{3}}-1)}(1-\epsilon^{\frac{4-\sigma}{3}}),$$

$$\mathrm{A}_4 = \frac{\mathrm{A}_2}{6}\frac{(\sigma-1)(\sigma-2)}{(\epsilon^{\frac{5}{3}}-1)}(\epsilon^{\frac{5}{3}}-\epsilon^{\frac{4-\sigma}{3}}), \tag{13}$$

where σ is an unknown parameter which will be used to minimize the rate of energy dissipation. For the trial stream function assumed here, the second invariant of the rate-of-deformation tensor in a dimensionless form is evaluated as

$$\overline{\mathrm{II}} = 6z^2\left[\mathrm{A}_2(\sigma-2)x^{\sigma-3} - \frac{3\mathrm{A}_3}{x^4} + 2\mathrm{A}_4 x\right]^2 + \frac{(1-z^2)}{2}\left[\mathrm{A}_2(\sigma-1)(\sigma-2)x^{\sigma-3} + \frac{6\mathrm{A}_3}{x^4} + 6\mathrm{A}_4 x\right]^2. \tag{14}$$

From a macroscopic mechanical energy balance, it can be easily shown that

$$UF_d = \int_V \mathrm{tr}(\boldsymbol{\tau}\cdot\nabla\mathbf{V})\,\mathrm{d}V, \tag{15}$$

where F_d is the drag force experienced by the swarm of bubbles.

An individual bubble is bounded by the spherical surface $x = 1$, with the normal pointing radially outward; and the surrounding liquid is bounded by the hypothetical envelope of radius $x = \epsilon^{-\frac{1}{3}}$, with the normal pointing inward. If $\tau_{r\theta} = 0$ at $x = 1$ and at $x = \epsilon^{-\frac{1}{3}}$, and $v_r = 0$ at $x = 1$, the surface integral appearing in [7] vanishes. Under these conditions [9], [10] and [15] can be combined, and the non-dimensionalization yields

$$X = \frac{C_D \mathrm{Re}}{24} \leqslant X_{UB} = \min\left\{\frac{2}{3E^2(n+1)}\int_{-1}^{1}\int_{\epsilon^{1/3}}^{1}\left[\left(1+2E^2\overline{\Pi}^*\right)^{\frac{n+1}{2}}-1\right]y^{-4}\,\mathrm{d}y\,\mathrm{d}z\right\}, \qquad [16]$$

where X is known as the drag correction factor, to be multiplied by the Stokes drag (for a rigid sphere) to obtain the drag coefficient of a swarm; C_D is the drag coefficient ($= F_d/\frac{1}{2}\rho U^2 \cdot \pi R^2$); Re is the Reynolds number, defined as $2\rho UR/\eta_0$; X_{UB} is the upper bound on X; E is the dynamic parameter arising from the non-dimensionalization of the governing equations, and is defined as $\lambda U/R$; y is the reciprocal radial coordinate $(1/x)$; and z denotes $\cos\theta$. Thus the minimization of the r.h.s. of [16] yields the value of X_{UB}.

LOWER BOUND CALCULATIONS

The calculation of the lower bound on X requires the knowledge of the components of the extra stress tensor which satisfy Cauchy's first law and prescribed boundary conditions. We choose here the same stress components as used by Chhabra & Dhingra (1986) for analysing the creeping flow around a liquid drop:

$$\tau_{rr} = -(\mathrm{C}y^{\mathrm{D}} + \mathrm{C}'y^{\mathrm{B}})z\left(\frac{\eta_0 U}{R}\right),$$

$$\tau_{\theta\theta} = -(\mathrm{F}y^{\mathrm{D}} + \mathrm{F}'y^{\mathrm{B}})z\left(\frac{\eta_0 U}{R}\right),$$

$$\tau_{\phi\phi} = -(\mathrm{G}y^{\mathrm{D}} + \mathrm{G}'y^{\mathrm{B}})z\left(\frac{\eta_0 U}{R}\right),$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mathrm{A}y^{\mathrm{B}}(1-z^2)^{\frac{1}{2}}\left(\frac{\eta_0 U}{R}\right),$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \tau_{r\phi} = \tau_{\phi r} = 0, \qquad [17]$$

where A, B, C, C′, D, F, F′, G and G′ are unknown constants to be evaluated from the prescribed boundary conditions and other considerations.

Following the considerations advanced by Mohan & Venkateswarlu (1976) and subsequently used by Chhabra & Raman (1984) and Chhabra & Dhingra (1986), and taking note of the fact that $\tau_{r\theta} = 0$ at the bubble surface ($x = 1/y = 1$) and at the free surface ($x = 1/y = \epsilon^{-\frac{1}{3}}$), it is easily shown that

$$\mathrm{A} = 0, \quad \mathrm{B} = 4, \quad \mathrm{C}' = \mathrm{G}' = \mathrm{F}' = 0, \quad \mathrm{D} = 2, \quad \mathrm{F} = \mathrm{G} = -\frac{\mathrm{C}}{2}. \qquad [18]$$

Hence, all the constants appearing in [17] are expressible in terms of C. Furthermore, the surface integral appearing in the stress variational principle [8] is evaluated to be equal to $-2\pi R\eta_0 U^2\mathrm{C}$. Now combining [8], [9], [11] and [18], and introducing the dimensionless variables, one obtains:

$$X = \frac{C_D \mathrm{Re}}{24} \geqslant X_{LB} = \max\left(-\frac{(n+1)}{3}\left[\left[\int_{\epsilon^{1/3}}^{1}\int_{-1}^{1}\left\{\left(1+2E^2\overline{\Pi}\right)^{\frac{n-1}{2}}\left(2\overline{\Pi}\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left. -\frac{1}{E^2(n+1)}\left[\left(1+2E^2\overline{\Pi}\right)^{\frac{n+1}{2}}-1\right]\right\}y^{-4}\,\mathrm{d}y\,\mathrm{d}z + \mathrm{C}\right]\right]\right). \qquad [19]$$

Thus the maximum value of the r.h.s. of [19] yields X_{LB}, which represents the lower limit of X for given values of n, E and ϵ. However, the evaluation of the function in [19] requires the value of $2\overline{\Pi}$. This is obtained from the following relation:

$$\overline{\Pi}_\tau = 4\,(1 + 2E^2\overline{\Pi})^{n-1}\,\overline{\Pi}, \qquad [20]$$

where $\overline{\Pi}_\tau$ is the dimensionless second invariant of the extra stress tensor. For the assumed stress profile, given by [17], $\overline{\mathrm{II}}_\tau$ is given by

$$\overline{\mathrm{II}}_\tau = \frac{\mathrm{II}_\tau}{\left(\dfrac{\eta_0 U}{R}\right)^2} = \bar{\tau}_{rr}^2 + \bar{\tau}_{\theta\theta}^2 + \bar{\tau}_{\phi\phi}^2 + 2\bar{\tau}_{r\theta}^2 = \frac{3C^2}{2} y^4 z^2. \qquad [21]$$

Thus [16] and [19] are the two final working equations.

It is worthwhile to point out here that, although in this work the drag correction factor has been defined with reference to the drag force experienced by a single rigid sphere, other definitions of the drag correction factor are also possible. For instance, one obvious choice is to define X as a deviation from the drag force experienced by a swarm of bubbles ascending in a Newtonian liquid (Gal-Or & Waslo 1968) under otherwise identical conditions. But it can readily be shown that the latter definition of X is simply related to the one used in this work via a function of gas holdup. Thus, it is a straightforward matter to convert the results from one form to another.

RESULTS AND DISCUSSION

The upper bound X_{UB} was obtained by minimizing the r.h.s. of [16], while the lower bound X_{LB} was obtained by maximizing the r.h.s. of [19]. The integrals appearing in [16] and [19] were evaluated numerically using the two-dimensional Simpson quadrature formula. Twenty steps in the radial direction and 40 steps in the θ direction were found to be adequate; further increases in the number of steps changed the results by $<0.1\%$. The Golden section search method was used to extremize the r.h.s.s of [16] and [19], respectively.

The values of the upper and lower bounds on the drag correction factor (X) were obtained over wide ranges of dimensionless material (n) and dynamic parameters (E, ϵ), as given below:

$$0.1 \leqslant n \leqslant 1.0; \quad 0.005 \leqslant E \leqslant 500; \quad 0 \leqslant \epsilon \leqslant 0.7.$$

Before embarking upon the presentation of the new theoretical results obtained in this study, it is instructive to examine the physical significance of each of the dimensionless variables. The Carreau model fluid behaviour index (n) is simply a measure of the extent of shear thinning behaviour. The smaller the value of n, the greater is the rate of decrease of apparent viscosity with the increasing shear rate. The dimensionless dynamic parameter E combines the fluid property (λ) and the kinematic condition as $\lambda U/R$, which can be re-written in the form of a ratio of a fluid characteristic time and a time scale of the process as $\lambda/R/U$. Hence, increasing values of E denote increasing importance of non-Newtonian effects. Finally, the gas holdup, ϵ, is an operating variable indicating the average gas content of the dispersion; the higher the value of ϵ, the stronger is the inter-bubble interaction.

The accuracy of the numerical results was checked by considering two limiting cases:

(i) When the gas holdup (ϵ) is zero, Happel's free surface cell model corresponds to the case of a single spherical bubble surrounded by a pool of liquid. Under these conditions both the upper and lower bounds on the drag correction (X) reduce to the results for a single bubble over the whole range of sionless parameters n and E, as reported by Chhabra & Dhingra (1986). comparisons between various theories and experiments have also been reported elsewhere (Chhabra & Dhingra 1986).

(ii) The second limiting case is one in which the liquid phase is Newtonian, i.e. or $\lambda = 0$ or both. Under these conditions, the present upper and lower bounds reduce to the Newtonian limit, as given by Gal-Or & Waslo (1968), over the entire range of ϵ studied in this case.

The upper and lower bounds (X_{UB} and X_{LB}) on the drag correction factor X are plotted in figures 2–4 for three different values of the Carreau model fluid behaviour index n (0.7, 0.5, 0.1), which cover the conditions of moderate to high levels of shear thinning. An examination of these figures reveals that the drag correction factor is always greater than that experienced by a single bubble moving in a non-Newtonian fluid. This augmentation in drag force leads to a reduction in the rise velocity of a swarm of bubbles, as compared with the case of a single bubble. Furthermore, as

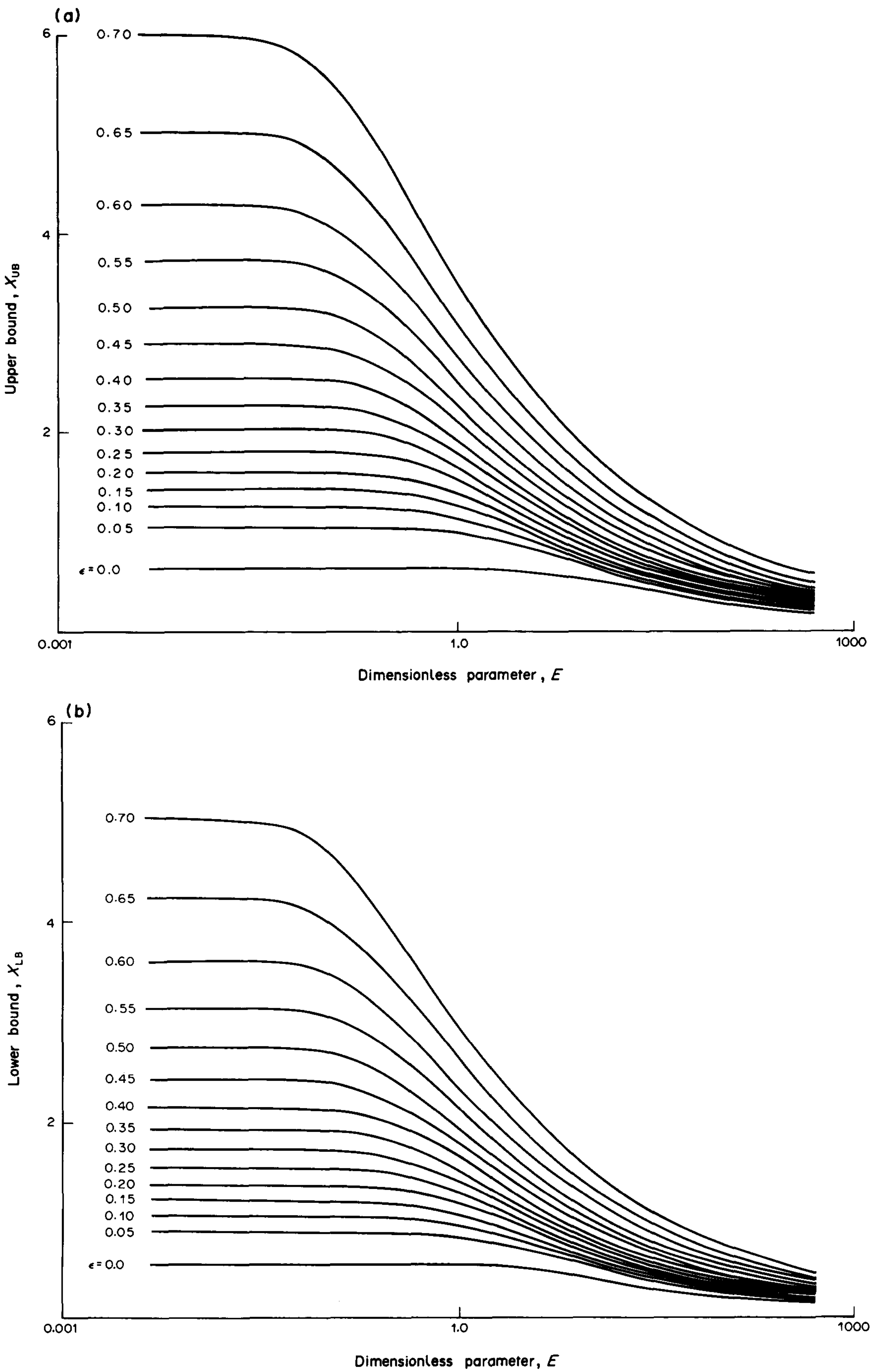

Figure 2. Upper (a) and lower (b) bounds on the drag correction factor for $n = 0.70$.

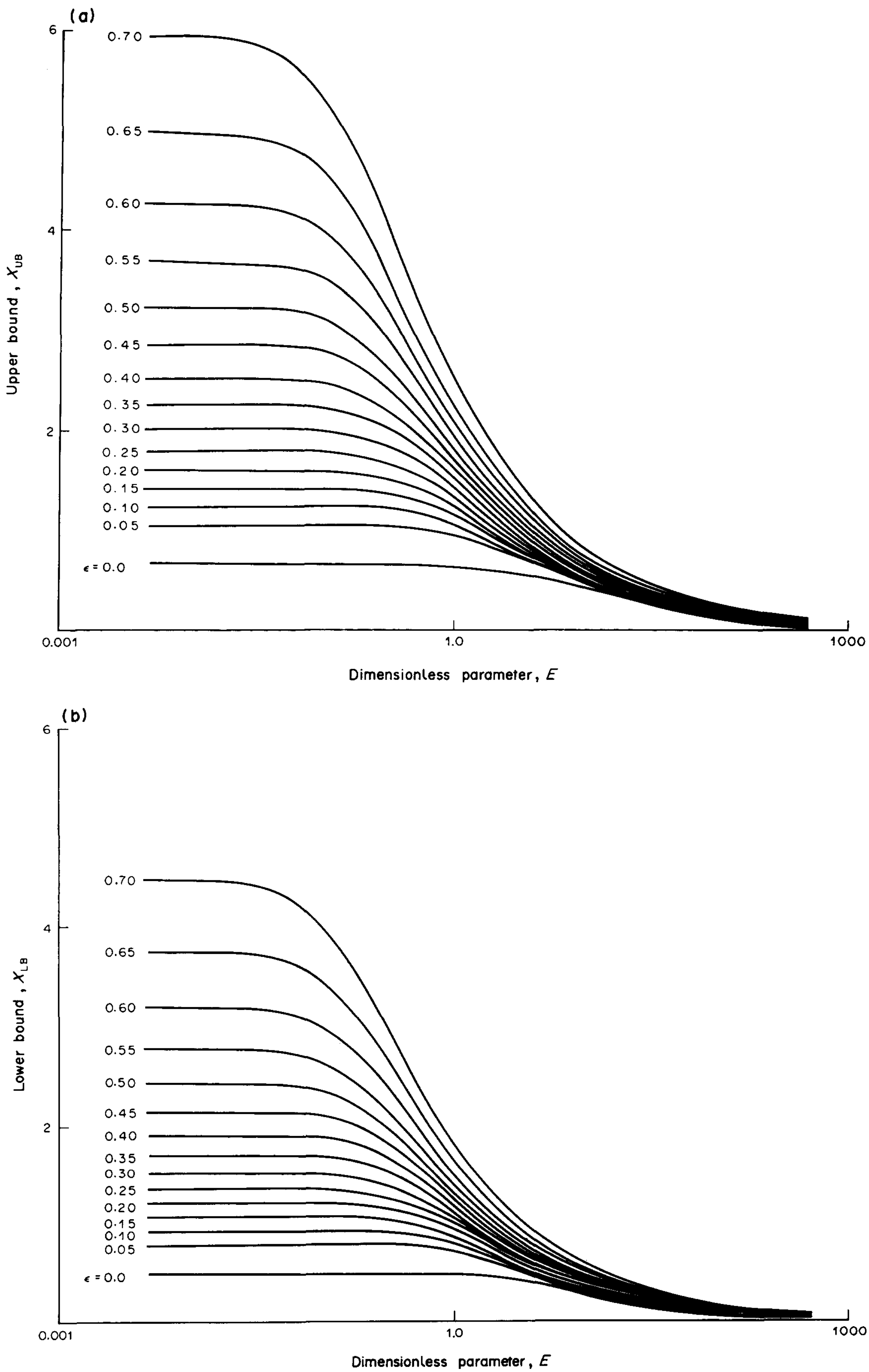

Figure 3. Upper (a) and lower (b) bounds on the drag correction factor for $n = 0.50$.

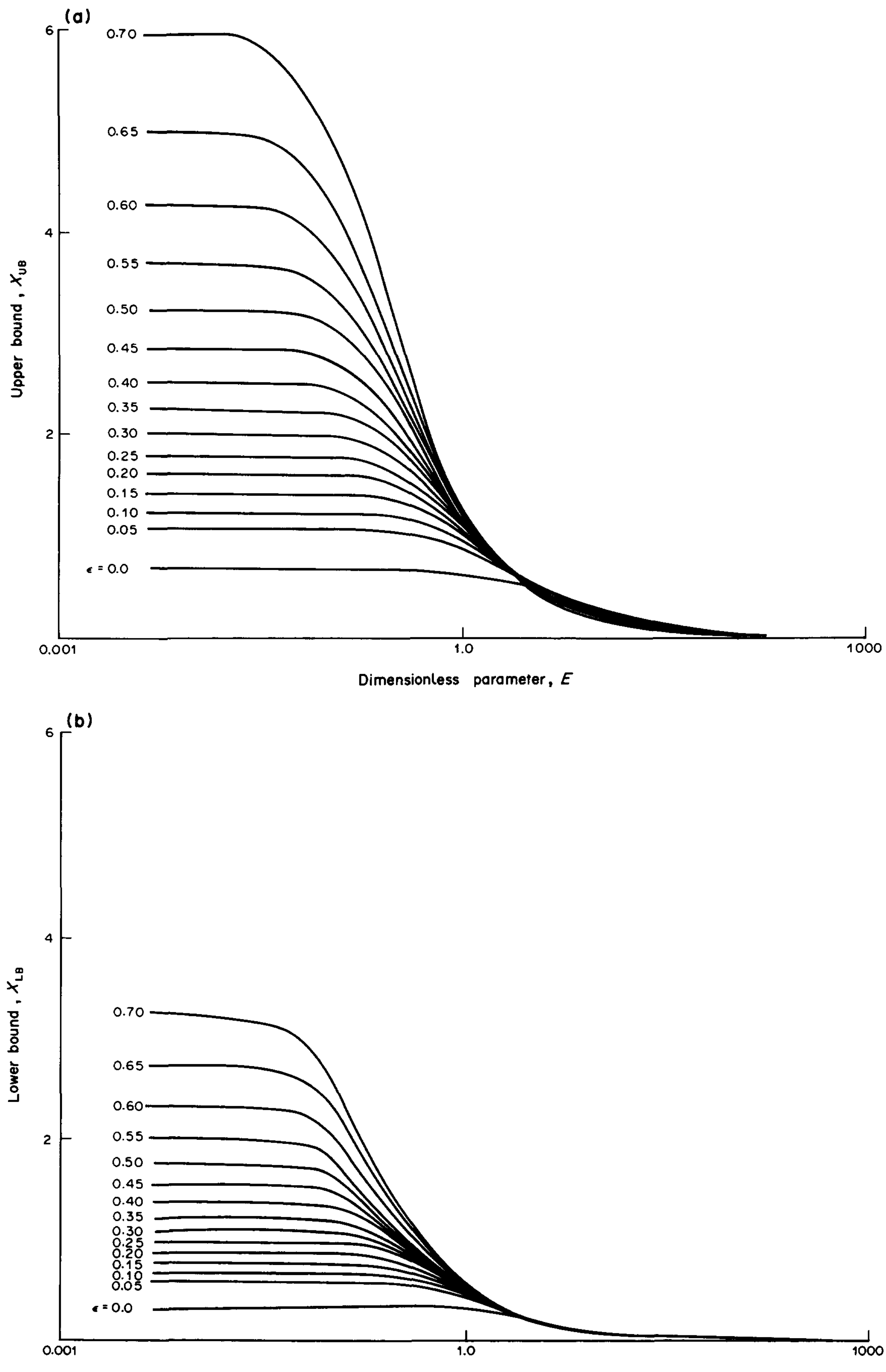

Figure 4. Upper (a) and lower (b) bounds on the drag correction factor for $n = 0.10$.

expected, for all values of n, the upper bound approaches the Newtonian value, i.e.

$$X = \frac{2}{3(1-\epsilon^{\frac{1}{3}})},$$

as the dimensionless parameter E is reduced to sufficiently small values. On the other hand, the lower bound reaches a limiting value which is $(n + 1)/2$ times the Newtonian value. The reasons for this behaviour are not at all clear but similar results have been reported by other workers (Hopke & Slattery 1970; Mohan & Venkateswarlu 1976). Further inspection of figures 2–4 suggests that the lower the value of n, the lower is the value of E at which the two bounds begin to deviate from the Newtonian value. The two bounds are reasonably close at low and high values of the dimensionless parameter E but show appreciable divergence from each other (though the maximum divergence is only about 40%) in the intermediate range of E ($\sim$0.5 to 1). This behaviour is much more clearly seen in figures 5 and 6, where the two bounds have been plotted for two different values of E.

From a physical point of view, the increasing amount of gas holdup (or contents) in a dispersion has two implications:

(i) It increases the number density of bubbles, which tends to cause an enhance- in the drag force experienced by a swarm.

(ii) As a result of that increase in the number density of bubbles, the liquid film entrapped between bubbles becomes thinner and, consequently, the liquid is subjected to higher levels of shearing action, which leads to a reduction in the effective viscosity of the liquid phase. This will allow an increase in the rise velocity of a swarm. Since, it is not possible to calculate the effective rates of shear, the actual decrease in effective viscosity (as a result of more vigorous shearing) cannot be estimated, and it would depend upon the values of n and E. However, it is evident that for given values of n, E and ϵ, the drag force terminal velocity) is determined by the relative magnitudes of the two opposing effects mentioned in the foregoing. Intuitively, it appears that for the condi- corresponding to a mild degree of shear thinning (e.g. the values of n being too different from unity, and small values of E), the increase in drag force due to the increased hindrance should far outweigh the reduction due to the shear thinning considerations, consequently, the rise velocity of a swarm should monotonically decrease with the gradual increase in gas holdup. This conjec- is supported by the behaviour observed in the case of swarms of bubbles ascending in Newtonian liquids wherein the rise velocity of a swarm decreases monotonically as the gas holdup increases. On the other hand, in the case of highly shear thinning conditions ($n \leqslant 0.5$ or so and moderate to high values dimensionless parameter E), the reduction in drag due to enhanced shearing likely to dominate over the increase in the drag due to the "crowding" effect. The net result being that the rise velocity of a swarm would increase. This possibly is one of the reasons for the "cross-over behaviour" seen in figure 4, whereas no such trend is present in figures 2 and 3 where the conditions correspond to a smaller degree of shear thinning behaviour.

Finally, the form of the Carreau viscosity equation suggests that at high values of shear rate ($\sqrt{2\mathrm{II}} >>> 1$), [5] reduces to the usual two-parameter power law model:

$$\eta = (\eta_0 \lambda^2)^{\frac{n-1}{2}} (2\mathrm{II})^{\frac{n-1}{2}}. \qquad [22]$$

Therefore, one would expect the upper and lower bounds, relating to sufficiently high values of the dimensionless parameter E, to approach the corresponding values obtained using the power law fluid model. Indeed this is so both for single bubbles as well as for swarms of bubbles moving through power law liquids. This behaviour occurs for $E > 500$ or so, and the results so obtained are in perfect agreement with the values reported in the literature for power law liquids (e.g. Chhabra & Dhingra 1986; Gummalam 1986).

From a practical point of view, if the rheological properties of the liquid phase (η_0, λ, n), gas

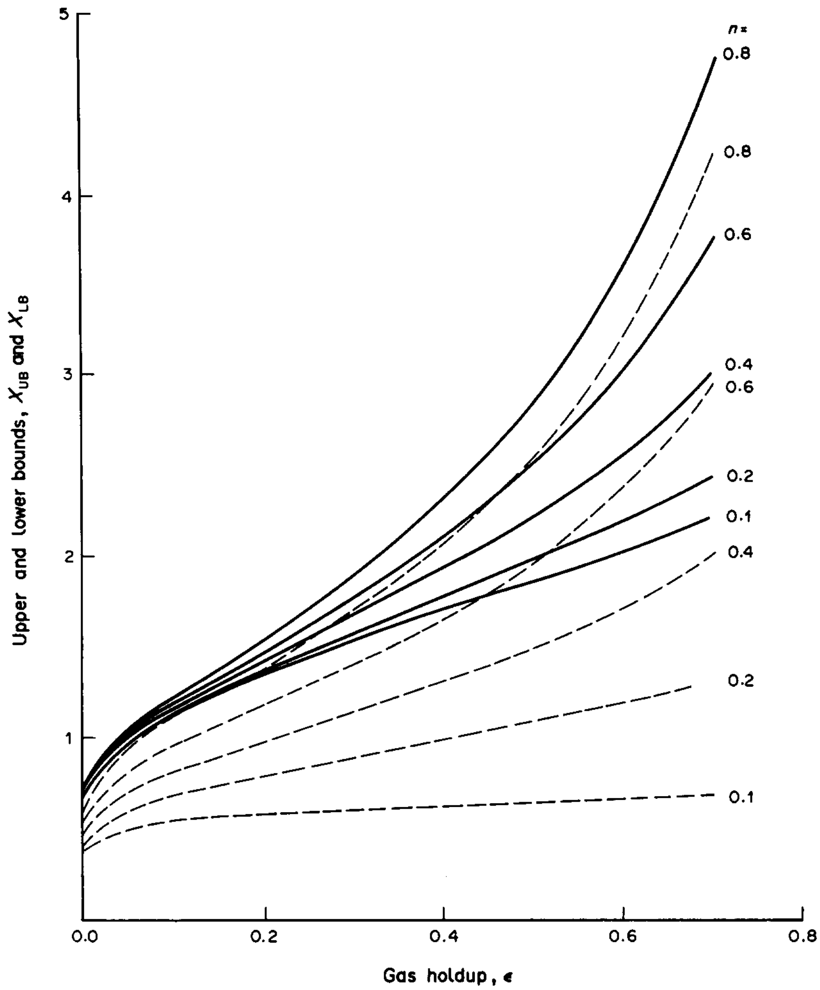

Figure 5. Effect of gas holdup on the drag correction factor for $E = 0.50$: ——, upper bound; ----, lower bound.

holdup (ϵ) and bubble size (R) are known, the rise velocity of a swarm can be estimated using the results reported herein. The upper and lower bounds obtained in this study can be used to calculate the maximum and minimum values of the rise velocity of a swarm in a given application. Since the location of the exact solution is not known, and in the absence of any other definite information available, as a first approximation the use of the arithmetic average of the two bounds is suggested. Unfortunately, no suitable experimental data on swarms of bubbles in non-Newtonian media are available to validate the predictions presented in this paper.

CONCLUSIONS

Using a combination of the variational principles, and Happel's free surface cell model, approximate results on the rise velocity of swarms of monosized spherical bubbles ascending through a quiescent pool of Carreau model liquids have been obtained. This investigation covers wide ranges of rheological behaviour ($0.1 \leqslant n \leqslant 1.0$ and $0.005 \leqslant E \leqslant 500$) and gas holdup ($0 \leqslant \epsilon \leqslant 0.70$). From the knowledge of the bubble size, gas holdup and the liquid rheology, the present theory permits the prediction of lower and upper bounds on the rise velocity of a swarm of bubbles which, in turn, facilitates the calculation of contact time of a gas in gas–liquid contacting devices such as bubble columns.

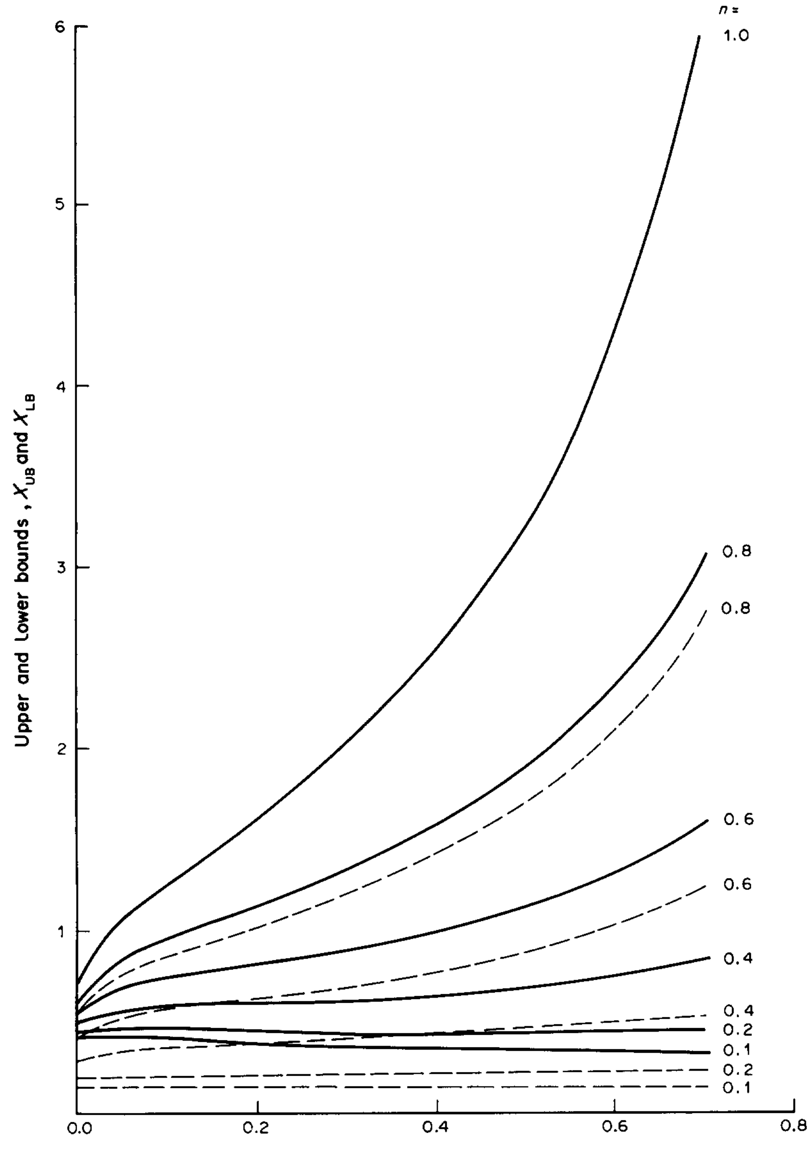

Figure 6. Effect of gas holdup on the drag correction factor for $E = 5.0$: ——, upper bound; ----, lower bound.

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